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J. Math. Anal. Appl. 321 (2006) 223–241

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Conditionally positive definite dot product kernels

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Received 28 April 2005

Available online 8 September 2005

Submitted by R.H. Torres

Abstract

This paper deals with conditionally positive definite kernels on Euclidean spaces. The focus here is on dot product kernels, that is, those depending on the inner product of the variables. Among the results, we include some properties relating conditional positive definiteness and standard convolution in the line and also results related to the characterization of the conditionally positive definite dot product kernels with respect to finite-dimensional polynomial spaces. We also introduce and characterize two large classes of strictly conditionally positive definite dot product kernels.

Keywords: Conditionally positive definite; Dot product kernels; Positive definite; Strictly conditionally positive definite

1. Introduction

Let A be a nonempty subset of $\mathbb{R}^m$ and $\mathcal{P}$ a subspace of the space Π^m of all polynomials in m variables with real coefficients. A kernel $(x, y) \in A \times A \mapsto f(x, y) \in \mathbb{R}$ is termed *conditionally positive definite with respect to* $\mathcal{P}$ if it is symmetric (that is, $f(x, y) = f(y, x)$, $x, y \in A$) and

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[1] Partially supported by PROCAD-CAPES, Grant No. 0092/01-0.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.08.024

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j f(x_i, x_j) \geqslant 0, \tag{1.1}$$

when $n \geqslant 1$, $x_1, x_2, \ldots, x_n$ are points in A and $(c_1, c_2, \ldots, c_n)$ is a vector in $\mathbb{R}^n$ such that

$$\sum_{i=1}^{n} c_i p(x_i) = 0, \quad p \in \mathcal{P}. \tag{1.2}$$

It is *strictly conditionally positive definite with respect to* $\mathcal{P}$ if it is conditionally positive definite with respect to $\mathcal{P}$ and strict inequality occurs in (1.1) when $n \geqslant 1$, $x_1, x_2, \ldots, x_n$ are distinct points in A, $(c_1, c_2, \ldots, c_n)$ is nonzero and (1.2) holds. If $\mathcal{P} = \{0\}$, the above concepts reduce themselves to those of positive definiteness and strict positive definiteness, respectively. If A is finite the definition requires that n be at most the cardinality of A.

The most interesting cases of the above concepts occur when the spaces are either finite-dimensional or *homogeneous*, that is, generated by homogeneous polynomials.

Conditionally positive definite kernels have been investigated in many contexts such as approximation theory, learning theory, etc. In recent years they have been used as a standard tool to generate solutions to certain interpolation problems in $\mathbb{R}^m$. Indeed, if the kernel $(x, y) \in A \times A \mapsto f(x, y)$ is strictly conditionally positive definite with respect to a finite-dimensional space $\mathcal{P}$ and the x_j are distinct then the interpolation problem

$$\sum_{j=1}^{n} c_j f(x_i, x_j) + q(x_i) = \lambda_i, \quad \lambda_i \in \mathbb{R},\ i = 1, 2, \ldots, n,\ q \in \mathcal{P}, \tag{1.3}$$

under condition (1.2) is always uniquely soluble as long as $p = 0$ is the only element of $\mathcal{P}$ vanishing at the interpolation points. We refer the reader to [1,3,9,10,15,16] and references therein for some information on conditionally positive definite kernels and papers authored by either Xingping Sun or F. Narcowich and J. Ward for some information on radial conditionally positive definite kernels. See also Micchelli's famous paper [12].

In this paper, we are mainly concerned with *conditionally positive definite dot product kernels*, that is, kernels of the form $f(x, y) = g(x \cdot y)$, $x, y \in A$, for some function g, in which $\cdot$ stands for the dot product of $\mathbb{R}^m$ (we will just write $x \cdot y = xy$ when $m = 1$). The first motivating result was the following description of the conditionally positive definite dot product kernels on $\mathbb{R}^m$, $m \geqslant 2$, given in [8]: a kernel $(x, y) \in \mathbb{R}^m \times \mathbb{R}^m \mapsto f(x \cdot y)$ is conditionally positive definite with respect to $\{0\}$ if and only if f is an everywhere convergent series of the form

$$f(t) = \sum_{k=0}^{\infty} a_k(f) t^k, \quad a_k(f) \geqslant 0, \tag{1.4}$$

that is, f is real entire and $f^{(k)}(0) \geqslant 0$, $k = 0, 1, \ldots$. The result also holds when we replace $\mathbb{R}^m$ with an infinite-dimensional Hilbert space while a similar characterization in the case $m = 1$ is not available yet. The second one came from a paper of Pinkus [13]. Beginning with a general dot product kernel as in (1.4), he found necessary and sufficient conditions for the strict conditional positive definiteness of the kernel on the space or on a symmetric and infinite subset. Precisely, given a function as in (1.4) and an infinite and symmetric subset A of $\mathbb{R}^m$, he proved that $(x, y) \in A \times A \mapsto f(x \cdot y)$ is strictly conditionally positive definite with respect to $\{0\}$ if and only if the set $\{k\colon a_k(f) > 0\}$ contains the index 0 plus an infinite number of even integers and an infinite number of odd integers. In addition, it was shown that the condition $0 \in \{k\colon a_k(f) > 0\}$ could be dropped when $0 \notin A$.

This paper presents results of the above type for nontrivial polynomial spaces. In the next two sections, we use some techniques we adapted from [8,13] to prove several results that converges to a similar characterization for conditionally positive definite kernels with respect to some finite-dimensional polynomial spaces (Theorem 3.13). Based upon these results, we analyze strict conditional positive definiteness in the last two sections of the paper.

The reader is advised that our results are not conclusive in the sense that they do not cover all polynomial spaces, so additional research will be needed to close the question. Many results in the paper are extensions or generalizations of results in [8,13]. As so, some techniques used here may be not new to readers acquainted with the results in those two papers. Reference [11] deals with a different topic but uses some of the same techniques.

2. Conditional positive definiteness via convolution

In this section, we study the effect of convolution over a function that generates a conditionally positive definite kernel. The results can be used to produce conditionally positive definite kernels with respect to a space $\mathcal{Q}$ from a given conditionally positive definite kernel with respect to another space $\mathcal{P}$. The required connection between $\mathcal{P}$ and $\mathcal{Q}$ will be made clear ahead.

We begin with some notation. For positive integers m and l we write the elements of $\mathbb{R}^{m+l}$ in the form (x, y), $x \in \mathbb{R}^m$, $y \in \mathbb{R}^l$. If p is an element of Π^m and q is an element of Π^l, we call

$$q(y)p(x), \quad x \in \mathbb{R}^m,\ y \in \mathbb{R}^l, \tag{2.1}$$

a *dilation* of p. If $\mathcal{B}$ is a subset of Π^m, a subspace $\mathcal{Q}$ of Π^{m+l} is called a *$\mathcal{B}$-dilation* when every element of $\mathcal{Q}$ is a linear combination of dilations of elements of $\mathcal{B}$, i.e., every element of $\mathcal{Q}$ is of the form

$$\sum_{i=1}^{k} q_i(y)p_i(x), \quad q_1, q_2, \ldots, q_k \in \Pi^l,\ p_1, p_2, \ldots, p_k \in \mathcal{B}. \tag{2.2}$$

Example ($m + l = 3$). If $\mathcal{B} = \{1, x, y\}$ then $[\{xz^2\}]$, $[\{1, x, xz, xz^2\}]$ and $[\{1, x, xz, x + xz^2\}]$ are $\mathcal{B}$-dilations while $[\{1, x, y, x^2, z\}]$ is not.

If $g : \mathbb{R} \mapsto \mathbb{R}$ is a function and $c \in \mathbb{R}$, we write g_c to denote the function $g_c(t) = g(t + c)$, $t \in \mathbb{R}$. Theorem 2.1 below has to do with the conditional positive definiteness of g_c with respect to $\mathcal{P}$ when g is conditionally positive definite with respect to a $\mathcal{P}$-dilation.

Theorem 2.1. *Let A be a subset of $\mathbb{R}^m$ and J a subset of $\mathbb{R}^l$. Let $\mathcal{B}$ be a subset of Π^m and $\mathcal{Q} \subset \Pi^{m+l}$ a $\mathcal{B}$-dilation. Let $c \in J$ and set $d := c \cdot c$. If $(x, y) \in (A \times J) \times (A \times J) \mapsto g(x \cdot y)$ is conditionally positive definite with respect to $\mathcal{Q}$ then $(x, y) \in A \times A \mapsto g_d(x \cdot y)$ is conditionally positive definite with respect to $[\mathcal{B}]$.*

Proof. Let $x_1, x_2, \ldots, x_n$ be points in A and $c_1, c_2, \ldots, c_n$ in $\mathbb{R}$ such that $\sum_{i=1}^{n} c_i p(x_i) = 0$, $p \in [\mathcal{B}]$. Define $y_j = (x_j, c)$, $j = 1, 2, \ldots, n$. Then $y_j \in A \times J$, $j = 1, 2, \ldots, n$, and

$$y_i \cdot y_j = x_i \cdot x_j + d, \quad i, j = 1, 2, \ldots, n. \tag{2.3}$$

In addition,

$$\sum_{i=1}^{n} c_i (qp)(y_i) = \sum_{i=1}^{n} c_i q(c) p(x_i) = q(c) \sum_{i=1}^{n} c_i p(x_i) = 0, \quad q \in \Pi^l,\ p \in \mathcal{B}, \tag{2.4}$$

and, consequently,

$$\sum_{i=1}^{n} c_i q(y_i) = 0, \quad q \in \mathcal{Q}. \tag{2.5}$$

If $(x, y) \in (A \times J) \times (A \times J) \mapsto g(x \cdot y)$ is conditionally positive definite with respect to $\mathcal{Q}$, it follows that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j g_d(x_i \cdot x_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j g(x_i \cdot x_j + d) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j g(y_i \cdot y_j) \geqslant 0. \tag{2.6}$$

This completes the proof. □

For strict conditional positive definiteness, the above theorem reads like this.

Theorem 2.2. *Let A be a subset of $\mathbb{R}^m$ and J a subset of $\mathbb{R}^l$. Let $\mathcal{B}$ be a subset of Π^m and $\mathcal{Q} \subset \Pi^{m+l}$ a $\mathcal{B}$-dilation. Let $c \in J$ and set $d := c \cdot c$. If $(x, y) \in (A \times J) \times (A \times J) \mapsto g(x \cdot y)$ is strictly conditionally positive definite with respect to $\mathcal{Q}$ then $(x, y) \in A \times A \mapsto g_d(x \cdot y)$ is strictly conditionally positive definite with respect to* $[\mathcal{B}]$.

Proof. Left to the reader. □

Toward the statement and proof of the next results we introduce convolution and some of its properties. If φ and g are measurable functions on $\mathbb{R}$, the *convolution* of φ and g is the function $\varphi * g$ defined by

$$(\varphi * g)(t) := \int_{-\infty}^{\infty} \varphi(t - s) g(s)\, ds, \quad t \in \mathbb{R}. \tag{2.7}$$

If the integrals in question exist, we always have $\varphi * g = g * \varphi$. To make sure that the convolution $\varphi * g$ is defined everywhere and some additional properties hold, the following assumptions will be in force: g is at least piecewise continuous, φ is $\mathbb{C}^\infty$ and nonnegative and the support $\operatorname{supp}(\varphi)$ of φ is a compact subset of $(-\infty, 0)$. The reader is advised that some of the results we prove ahead hold in fact, with a weaker set of hypotheses on φ. We do not want to discuss this matter here but inform that every property about convolution used in this paper is to be found in standard references such as [4,5]. Finally, we observe that some additional hypotheses on the function φ will be needed in some specific points ahead.

Theorem 2.3. *Let A be a subset of $\mathbb{R}^m$ and J a subset of $\mathbb{R}^l$. Let $\mathcal{B}$ be a subset of Π^m and $\mathcal{Q} \subset \Pi^{m+l}$ a $\mathcal{B}$-dilation. Let g be a function such that $(x, y) \in (A \times J) \times (A \times J) \mapsto g(x \cdot y)$ is conditionally positive definite with respect to $\mathcal{Q}$. If* $\operatorname{supp}(\varphi) \subset \{-c \cdot c\colon\ c \in J\}$ *then $(x, y) \in A \times A \mapsto (\varphi * g)(x \cdot y)$ is conditionally positive definite with respect to* $[\mathcal{B}]$.

Proof. Let $x_1, x_2, \ldots, x_n$ be points in A and $c_1, c_2, \ldots, c_n \in \mathbb{R}$. A simple change of variables leads to

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j (\varphi * g)(x_i \cdot x_j) &= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \int_{-\infty}^{\infty} \varphi(x_i \cdot x_j - s) g(s)\, ds \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \int_{-\infty}^{\infty} \varphi(-s) g(x_i \cdot x_j + s)\, ds,\end{aligned}$$

while further calculations lead to

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j (\varphi * g)(x_i \cdot x_j) &= \int_{-\infty}^{\infty} \varphi(-s) \left(\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j g(x_i \cdot x_j + s) \right) ds \\ &= \int_{-\operatorname{supp}(\varphi)} \varphi(-s) \left(\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j g_s(x_i \cdot x_j) \right) ds.\end{aligned}$$

If $\operatorname{supp}(\varphi) \subset \{-c \cdot c\colon c \in J\}$, the integration is in fact taking place on $\{c \cdot c\colon c \in J\}$. Since $(x, y) \in (A \times J) \times (A \times J) \mapsto g(x \cdot y)$ is conditionally positive definite with respect to $\mathcal{Q}$, Theorem 2.1 implies that g_s is conditionally positive definite with respect to $[\mathcal{B}]$, for every s in the subset. Thus, the integrand in the very last expression above is nonnegative when $\sum_{i=1}^{n} c_i p(x_i) = 0$, $p \in [\mathcal{B}]$. $\square$

Theorem 2.4. *Let A be a subset of $\mathbb{R}^m$ and J a subset of $\mathbb{R}^l$. Let $\mathcal{B}$ be a subset of Π^m and $\mathcal{Q} \subset \Pi^{m+l}$ a $\mathcal{B}$-dilation. Let g be a function such that the kernel $(x, y) \in (A \times J) \times (A \times J) \mapsto g(x \cdot y)$ is strictly conditionally positive definite with respect to $\mathcal{Q}$. If $\operatorname{supp}(\varphi)$ is a subset of $\{-c \cdot c\colon c \in J\}$ having a nonempty interior then $(x, y) \in A \times A \mapsto (\varphi * g)(x \cdot y)$ is strictly conditionally positive definite with respect to $[\mathcal{B}]$.*

Proof. Under the present assumptions and with the additional hypotheses on φ, Theorem 2.3 implies that the integrand we referred to in the previous proof is positive on some interval, when the additional assumption $\sum_{i=1}^{n} |c_i| > 0$ holds. The rest follows from this. $\square$

Corollary 2.5. *Let $\mathcal{B}$ be a subset of Π^m and $\mathcal{Q} \subset \Pi^{m+l}$ a $\mathcal{B}$-dilation. If g is a function such that $(x, y) \in \mathbb{R}^{m+l} \times \mathbb{R}^{m+l} \mapsto g(x \cdot y)$ is conditionally positive definite with respect to $\mathcal{Q}$ then $(x, y) \in \mathbb{R}^m \times \mathbb{R}^m \mapsto (\varphi * g)(x \cdot y)$ is conditionally positive definite with respect to $[\mathcal{B}]$. If $\operatorname{supp}(\varphi)$ has nonempty interior and $(x, y) \in \mathbb{R}^{m+l} \times \mathbb{R}^{m+l} \mapsto g(x \cdot y)$ is strictly conditionally positive definite with respect to $\mathcal{Q}$ then $(x, y) \in \mathbb{R}^m \times \mathbb{R}^m \mapsto (\varphi * g)(x \cdot y)$ is strictly conditionally positive definite with respect to $[\mathcal{B}]$.*

3. Conditional positive definiteness: General results

This section contains a variety of results. First, we fix a function differentiable up to a certain order and find necessary conditions under which the kernel generated by the function is conditionally positive definite with respect to a finite-dimensional polynomial space $\mathcal{P}$. As a consequence, we characterize the conditionally positive definite kernels with respect to some special finite-dimensional polynomial spaces, which are generated by a real entire function. The rest of the section deals with similar results, but bringing in the convolution ideas from the previous section. We make a connection between conditional positive definiteness and absolute monotonicity and show that under certain conditions a function generating a conditionally positive definite kernel with respect to a $\mathcal{B}$-dilation is necessarily real entire with nonnegative higher order derivatives at zero.

As we mentioned in the introduction, conditional positive definiteness with respect to $\{0\}$ was completely determined by Lu and Sun [8] in the case $m \geqslant 2$. As far as we know, the case $m = 1$ is still open and some elementary examples mentioned in [13] indicate that the final characterization in this case will not follow the pattern of the other cases.

We like to think that a final characterization of the conditionally positive definite kernels will lead to a similar series representation for the function generating the kernel but with a different condition on the coefficients, one that depends on the polynomial space $\mathcal{P}$. The results in this section ratify that this is in fact the case for many choices of the space.

The first result in the section is divided in two steps. We begin by analyzing the one-dimensional case, considering the higher-dimensional case in a second stage. Given a polynomial subspace $\mathcal{P}$ of Π^1, we use the symbol $\Gamma^1_{\mathcal{P}}$ to denote the set of all nonnegative integers k for which there is no polynomial in $\mathcal{P}$ containing the monomial t^k as a summand. In case $\mathcal{P}$ is finite-dimensional, the letter β will denote the highest degree of an element of $\mathcal{P}$, that is,

$$\beta := \max\bigl\{\deg(p)\colon\ p \in \mathcal{P}\bigr\}. \tag{3.1}$$

Theorem 3.1. *Let $\mathcal{P}$ be a finite-dimensional subspace of Π^1. If f is $\mathbb{C}^{\gamma+1}$ for some $\gamma \geqslant \beta + 1$ and $(x, y) \in \mathbb{R}\times\mathbb{R} \mapsto f(xy)$ is conditionally positive definite with respect to $\mathcal{P}$ then $f^{(k)}(0) \geqslant 0$ whenever $k \in \Gamma^1_{\mathcal{P}} \cap \{0, 1, \ldots, \gamma\}$.*

Proof. Let f be a function as in the statement of the theorem. Let $k \in \Gamma^1_{\mathcal{P}} \cap \{0, 1, \ldots, \gamma\}$ and choose an integer n in the following way: if $k \geqslant \beta + 1$ let $n = k$, otherwise, let $n = \beta + 1$. Next, choose $n + 1$ nonzero and distinct points $x_1, x_2, \ldots, x_{n+1}$ in $\mathbb{R}$ and take $c_1, c_2, \ldots, c_{n+1}$ in $\mathbb{R}$ in such a way that

$$\sum_{i=1}^{n+1} c_i x_i^j = \begin{cases} 0, & j \in \{0, 1, \ldots, n\} \setminus \{k\}, \\ 1, & j = k. \end{cases} \tag{3.2}$$

Pick a closed and symmetric interval I so that $\{x_i x_j\colon\ i, j = 1, 2, \ldots, n + 1\} \subset I$. Since $0 \in I$ and f has continuous derivatives up to order $n + 1$ in I, Taylor's theorem guarantees that

$$f(t) = f(0) + \sum_{\mu=1}^{n} \frac{f^{(\mu)}(0)}{\mu!} t^\mu + R_n(t), \quad t \in I, \tag{3.3}$$

in which

$$R_n(t) = \frac{1}{n!} \int_0^t f^{(n+1)}(x)(t - x)^n\, dx. \tag{3.4}$$

Let $\epsilon \in (0, 1)$. Since $\sum_{i=1}^{n+1} c_i p(\epsilon^{1/2} x_i) = 0$, $p \in \mathcal{P}$, the conditional positive definiteness of the kernel with respect to $\mathcal{P}$ yields

$$0 \leqslant \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} c_i c_j f(\epsilon x_i x_j)$$

$$= f(0)\left(\sum_{i=1}^{n+1} c_i\right)^2 + \sum_{\mu=1}^{n} \frac{f^{(\mu)}(0)}{\mu!}\epsilon^{\mu}\left(\sum_{i=1}^{n+1} c_i x_i^{\mu}\right)^2 + \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} c_i c_j R_n(\epsilon x_i x_j),$$

that is,

$$0 \leqslant f^{(k)}(0) + \frac{k!}{\epsilon^k}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} c_i c_j R_n(\epsilon x_i x_j). \tag{3.5}$$

Using the Lagrange form of R_n, we obtain

$$\begin{aligned}\left|R_n(\epsilon x_i x_j)\right| &\leqslant \max\left\{\left|f^{(n+1)}(x)\right|\colon x \in I\right\}\frac{|\epsilon x_i x_j|^{n+1}}{(n+1)!}\\ &= \frac{\epsilon^{n+1}}{(n+1)!}\max\left\{\left|f^{(n+1)}(x)\right|\colon x \in I\right\}|x_i x_j|^{n+1}, \quad i, j = 1, 2, \ldots, n+1,\end{aligned}$$

whence

$$\begin{aligned}&\frac{k!}{\epsilon^k}\left|\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} c_i c_j R_n(\epsilon x_i x_j)\right|\\ &\quad\leqslant \frac{\epsilon^{n+1-k}}{(n+1)n\cdots(k+1)}\max\left\{\left|f^{(n+1)}(x)\right|\colon x \in I\right\}\left(\sum_{i=1}^{n+1}|c_i||x_i|^{n+1}\right)^2.\end{aligned}$$

Since $n+1-k \geqslant 1$, letting $\epsilon \to 0^+$ in (3.5), we obtain $f^{(k)}(0) \geqslant 0$. $\square$

In order to establish a similar result in higher dimensions, we need to introduce a bit of notation. If $\mathcal{P}$ is a subspace of Π^m and $l \in \{1, 2, \ldots, m\}$ we write

$$\mathcal{P}_l := \left\{p(1, 1, \ldots, 1, t, 1, \ldots, 1)\colon p \in \mathcal{P}\right\}, \tag{3.6}$$

in which $(1, 1, \ldots, 1, t, 1, \ldots, 1)$ is the vector of $\mathbb{R}^m$ whose lth component is t and all the others are equal to 1. The set $\Gamma_{\mathcal{P}}^m$ is then the set of all nonnegative integers k for which there is an index $l \in \{1, 2, \ldots, m\}$ such that $\mathcal{P}_l$ does not contain a polynomial having t^k as a summand, that is, $\Gamma_{\mathcal{P}}^m = \bigcup_{l=1}^m \Gamma_{\mathcal{P}_l}^1$. From now on, the reader is advised that standard multi-index notation will be used.

Theorem 3.2. *Let $\mathcal{P}$ be a finite-dimensional subspace of Π^m. If f is $\mathbb{C}^{\gamma+1}$ for some $\gamma \geqslant \beta + 1$ and $(x, y) \in \mathbb{R}^m \times \mathbb{R}^m \mapsto f(x \cdot y)$ is conditionally positive definite with respect to $\mathcal{P}$ then $f^{(k)}(0) \geqslant 0$ when $k \in \Gamma_{\mathcal{P}}^m \cap \{0, 1, \ldots, \gamma\}$.*

Proof. We adapt the proof of Theorem 3.1 to the present context. Let f be a function as described in the statement of the theorem. Let $k \in \Gamma_{\mathcal{P}}^m \cap \{0, 1, \ldots, \gamma\}$ and pick l such that $k \in \Gamma_{\mathcal{P}_l}^1$. Set $n = \gamma$ and choose $n+1$ nonzero and distinct points $x_1, x_2, \ldots, x_{n+1}$ in $\mathbb{R}$ and scalars $c_1, c_2, \ldots, c_{n+1}$ in $\mathbb{R}$ as in the proof of Theorem 3.1. If $k \leqslant \beta$ define

$$y_i := (1, 1, \ldots, 1, x_i, 1, \ldots, 1) \in \mathbb{R}^m, \quad i = 1, 2, \ldots, n+1, \tag{3.7}$$

in which x_i is in the lth component of y_i. Otherwise, define

$$y_i := (x_i, 1, 1, \ldots, 1) \in \mathbb{R}^m, \quad i = 1, 2, \ldots, n+1. \tag{3.8}$$

Pick a closed and symmetric interval I such that $\{y_i \cdot y_j\colon\ i, j = 1, 2, \ldots, n+1\} \subset I$. Let $\epsilon \in (0,1)$. Since $\sum_{i=1}^{n+1} c_i p(\epsilon^{1/2} y_i) = 0,\ p \in \mathcal{P}$, the conditional positive definiteness of the kernel with respect to $\mathcal{P}$ yields

$$\begin{aligned}
0 &\leqslant \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} c_i c_j f(\epsilon y_i \cdot y_j)\\
&= \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} c_i c_j \left(f(0) + \sum_{\mu=1}^{n} \frac{f^{(\mu)}(0)}{\mu!} \epsilon^{\mu} (y_i \cdot y_j)^{\mu} + R_n(\epsilon y_i \cdot y_j) \right)\\
&= \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} c_i c_j \left(f(0) + \sum_{\mu=1}^{n} \frac{f^{(\mu)}(0)}{\mu!} \epsilon^{\mu} \sum_{\nu=0}^{\mu} \binom{\mu}{\nu} (x_i x_j)^{\nu} (m-1)^{\mu-\nu} + R_n(\epsilon y_i \cdot y_j) \right).
\end{aligned}$$

Defining $c_{\mu,\nu} := (m-1)^{\mu-\nu}/(\mu-\nu)!\nu!$, the above inequality reduces itself to

$$\begin{aligned}
0 &\leqslant f(0)\left(\sum_{i=1}^{n+1} c_i\right)^2 + \sum_{\mu=1}^{n} \epsilon^{\mu} f^{(\mu)}(0) \sum_{\nu=0}^{\mu} c_{\mu,\nu} \left(\sum_{i=1}^{n+1} c_i x_i^{\nu}\right)^2 + \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} c_i c_j R_n(\epsilon y_i \cdot y_j)\\
&= \sum_{\mu=k}^{n} c_{\mu,k} \epsilon^{\mu} f^{(\mu)}(0) + \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} c_i c_j R_n(\epsilon y_i \cdot y_j).
\end{aligned}$$

If $k = n$, the formula reduces itself to one very close to (3.5). If $k < n$, it reduces to

$$0 \leqslant f^{(k)}(0) + \sum_{\mu=k+1}^{n} k! c_{\mu,k} \epsilon^{\mu-k} f^{(\mu)}(0) + \frac{k!}{\epsilon^k} \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} c_i c_j R_n(\epsilon y_i \cdot y_j). \tag{3.9}$$

In any case, using the inequality

$$\begin{aligned}
&\left| \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} c_i c_j R_{n+1}(\epsilon y_i \cdot y_j) \right|\\
&\quad \leqslant \frac{\epsilon^{n+1}}{(n+1)!} \max\left\{ \left| f^{(n+1)}(x) \right|\colon\ x \in I \right\} \left(\sum_{i=1}^{n+1} |c_i| (y_i \cdot y_i)^{(n+1)/2} \right)^2
\end{aligned}$$

and the fact that $\mu - k \geqslant 1$ when $n > k$ and $\mu = k+1, k+2, \ldots, n$, we may let $\epsilon \to 0^+$ to obtain $f^{(k)}(0) \geqslant 0$. $\square$

Remarks. Theorem 3.2 remains valid when we replace $\mathbb{R}^m$ with an appropriate infinite subset. The case $m = 1$ with $\mathcal{P} = \{0\}$ generalizes Lemma 1 in [8].

A polynomial subspace $\mathcal{P}$ of Π^m is said to be *adequate* when the following condition holds: if $k \notin \Gamma_{\mathcal{P}}^m$ then $x^{\alpha} \in \mathcal{P}$, $|\alpha| = k$. For example, in the case $m = 2$, the spaces $[\{1, x, y, x^2, xy\}]$ and $[\{1, x, y, x^2, xy, y^2, x^5, x^4 y, x^3 y^2\}]$ are adequate while $[\{x^2, xy\}]$ is not.

Theorem 3.3. *Let $\mathcal{P}$ be a finite-dimensional adequate subspace of Π^m and let f be an everywhere convergent series of the form*

$$f(t) = \sum_{k=0}^{\infty} a_k(f) t^k. \tag{3.10}$$

Then $(x, y) \in \mathbb{R}^m \times \mathbb{R}^m \mapsto f(x \cdot y)$ is conditionally positive definite with respect to $\mathcal{P}$ if and only if $a_k(f) \geqslant 0$, $k \in \Gamma_{\mathcal{P}}^m$.

Proof. One implication follows from Theorem 3.2. As for the other, let $x_1, x_2, \ldots, x_n$ be points in $\mathbb{R}^m$ and let $c_1, c_2, \ldots, c_n$ be scalars satisfying (1.2). Then

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j f(x_i \cdot x_j) &= \sum_{k=0}^{\infty} a_k(f) \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \left(\sum_{|\alpha|=k} \frac{k!}{\alpha!} x_i^{\alpha} x_j^{\alpha}\right) \\ &= \sum_{k=0}^{\infty} a_k(f) \sum_{|\alpha|=k} \frac{k!}{\alpha!} \left(\sum_{i=1}^{n} c_i x_i^{\alpha}\right)^2.\end{aligned}$$

Due to the additional hypothesis we made on $\mathcal{P}$, it follows that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j f(x_i \cdot x_j) = \sum_{k \in \Gamma_{\mathcal{P}}^m} a_k(f) \sum_{|\alpha|=k} a_k(f) \frac{k!}{\alpha!} \left(\sum_{i=1}^{n} c_i x_i^{\alpha}\right)^2 \geqslant 0, \tag{3.11}$$

and this concludes the proof. □

We remark that Theorem 4 in [8] is a particular case of Theorem 3.3 above. We close the section introducing convolution into our results. The goal is to obtain a full characterization for conditional positive definiteness with respect to some finite-dimensional spaces. More notation regarding convolution is needed. Let φ and g as in Section 2. For $\epsilon > 0$, we set

$$\varphi_\epsilon(t) := \epsilon^{-1} \varphi\big(\epsilon^{-1} t\big), \quad t \in \mathbb{R}. \tag{3.12}$$

It is known that (see [4, p. 208]) if $\int_{-\infty}^{\infty} \varphi(s)\, ds = 1$ and g is continuous at x then

$$\lim_{\epsilon \to 0^+} (\varphi_\epsilon * g)(x) = g(x). \tag{3.13}$$

Theorem 3.4 below is an extension of Theorem 3.2. The trick involving convolution used in its proof is not new. The reader can find the very same reasoning in [6,8,17].

Theorem 3.4. *Let $\mathcal{B}$ be a finite subset of Π^m and $\mathcal{Q} \subset \Pi^{m+l}$ a $\mathcal{B}$-dilation. If f is $\mathbb{C}^\gamma$ for some γ and $(x, y) \in \mathbb{R}^{m+l} \times \mathbb{R}^{m+l} \mapsto f(x \cdot y)$ is conditionally positive definite with respect to $\mathcal{Q}$ then $f^{(k)}(0) \geqslant 0$, $k \in \Gamma_{[\mathcal{B}]}^m \cap \{0, 1, \ldots, \gamma\}$.*

Proof. Let f be as in the statement of the theorem. Pick a function φ as in Section 2 and satisfying $\int_{-\infty}^{\infty} \varphi(s)\, ds = 1$. Consider the functions φ_ϵ defined in (3.12). It is quite clear that the φ_ϵ have the same properties φ does. If $(x, y) \in \mathbb{R}^{m+l} \times \mathbb{R}^{m+l} \mapsto f(x \cdot y)$ is conditionally positive definite with respect to $\mathcal{Q}$, Corollary 2.5 implies that $(x, y) \in \mathbb{R}^m \times \mathbb{R}^m \mapsto (\varphi_\epsilon * f)(x \cdot y)$ is conditionally positive definite with respect to $[\mathcal{B}]$. We may apply Theorem 3.2 to conclude that

$$(\varphi_\epsilon * f)^{(k)}(0) \geqslant 0, \quad k \in \Gamma_{[\mathcal{B}]}^m. \tag{3.14}$$

In particular,

$$\big(\varphi_\epsilon * f^{(k)}\big)(0) \geqslant 0, \quad k \in \Gamma^m_{[\mathcal{B}]} \cap \{0, 1, \ldots, \gamma\}. \tag{3.15}$$

Letting $\epsilon \to 0^+$, we reach

$$f^{(k)}(0) \geqslant 0, \quad k \in \Gamma^m_{[\mathcal{B}]} \cap \{0, 1, \ldots, \gamma\}. \tag{3.16}$$

This completes the proof. □

For the sake of completeness, we register the following obvious corollary.

Corollary 3.5. *Let $\mathcal{B}$ be a finite subset of Π^m and $\mathcal{Q} \subset \Pi^{m+l}$ a $\mathcal{B}$-dilation. If f is $\mathbb{C}^\infty$ and $(x, y) \in \mathbb{R}^{m+l} \times \mathbb{R}^{m+l} \mapsto f(x \cdot y)$ is conditionally positive definite with respect to $\mathcal{Q}$ then $f^{(k)}(0) \geqslant 0$, $k \in \Gamma^m_{[\mathcal{B}]}$.*

Next, we recall the concept of absolute monotonicity. A function f is *absolutely monotonic* in the interval (a, b) if it has nonnegative derivatives of all orders there. It is absolutely monotonic in $[a, b)$ if it is absolutely monotonic in (a, b) and is continuous in $[a, b)$. An absolute monotonic function f on $[a, b)$ can be extended analytically to $\{z\colon |z - a| < b\}$. In particular, f has a series representation in the form

$$f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a^+)}{k!}(t - a)^k, \quad a \leqslant t < b, \tag{3.17}$$

in which $f^{(k)}(a^+)$ is the right derivative of order k of f at a. This result and the next one are to be found in [18, pp. 147–151].

Lemma 3.6. *A function f is absolutely monotonic in $[a, b)$ if and only if*

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(t + kh) \geqslant 0, \quad n = 0, 1, \ldots, \tag{3.18}$$

for all t and h such that $a \leqslant t < t + h < t + 2h < \cdots < t + nh < b$.

Theorem 3.7. *Let $\mathcal{B}$ be a finite subset of Π^m and $\mathcal{Q} \subset \Pi^{m+l}$ a $\mathcal{B}$-dilation. Put $\mathcal{P} = [\mathcal{B}]$ and let β be as before. If f is $\mathbb{C}^{\beta+1}$ and $(x, y) \in \mathbb{R}^{m+l} \times \mathbb{R}^{m+l} \mapsto f(x \cdot y)$ is conditionally positive definite with respect to $\mathcal{Q}$ then $f^{(\beta+1)}$ is absolutely monotonic in $[0, \infty)$. In particular, $f^{(k)}(0^+) \geqslant 0$, $k \geqslant \beta + 1$.*

Proof. Let f be as in the statement of the theorem and let φ and φ_ϵ as in the proof of Theorem 3.4. For each $c > 0$, the auxiliary function $\varphi_{\epsilon,c}(t) := \varphi_\epsilon(c + t)$ is $\mathbb{C}^\infty$ and $\operatorname{supp}(\varphi_{\epsilon,c}) = -c + \operatorname{supp}(\varphi_\epsilon) \subset (-\infty, 0)$. Thus, $\varphi_{\epsilon,c}$ and φ_ϵ have the same features. In particular, Corollary 2.5 implies that every kernel $(x, y) \in \mathbb{R}^m \times \mathbb{R}^m \mapsto (\varphi_{\epsilon,c} * f)(x \cdot y)$ is conditionally positive definite with respect to $\mathcal{P}$. Theorem 3.2 implies that

$$(\varphi_{\epsilon,c} * f)^{(k)}(0) \geqslant 0, \quad k \in \Gamma^m_{\mathcal{P}},\ c > 0, \tag{3.19}$$

or, equivalently,

$$(\varphi_\epsilon * f)^{(k)}(c) \geqslant 0, \quad k \in \Gamma^m_{\mathcal{P}},\ c > 0. \tag{3.20}$$

In particular, $(\varphi_\epsilon * f)^{(\beta+1)}$ is absolutely monotonic in $[0, \infty)$. Due to Lemma 3.6,

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\varphi_\epsilon * f)^{(\beta+1)}(t+jh) \geqslant 0, \quad k \geqslant 0,\ h>0,\ t \geqslant 0, \tag{3.21}$$

and, consequently,

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\big(\varphi_\epsilon * f^{(\beta+1)}\big)(t+jh) \geqslant 0, \quad k \geqslant 0,\ h>0,\ t \geqslant 0. \tag{3.22}$$

Taking the limit when $\epsilon \to 0^+$, we deduce that

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} f^{(\beta+1)}(t+jh) \geqslant 0, \quad k \geqslant 0,\ h>0,\ t \geqslant 0, \tag{3.23}$$

that is, $f^{(\beta+1)}$ is absolutely monotonic in $[0,\infty)$. The rest follows from the comments preceding Lemma 3.6. □

Next, we intend to refine the previous theorem, at least when the polynomial space is homogeneous. In order to do that we need a sequence of independent results.

Proposition 3.8. *The class of conditionally positive definite kernels with respect to a polynomial space $\mathcal{P}$ is closed under pointwise convergence.*

Proof. Left to the reader. □

Proposition 3.9. *Let $r \in \mathbb{R}$. If $\mathcal{P}$ is a homogeneous subspace of Π^m and $(x,y) \in \mathbb{R}^m \times \mathbb{R}^m \mapsto f(x \cdot y)$ is conditionally positive definite with respect to $\mathcal{P}$ then the same is true for the kernels $(x,y) \in \mathbb{R}^m \times \mathbb{R}^m \mapsto f_{r,\mu}(x \cdot y)$, $\mu = 1,2$, in which $f_{r,\mu}(t) := f(t) + f(r^2 t) + (-1)^\mu 2 f(rt)$, $t \in \mathbb{R}$.*

Proof. Let $x_1, x_2, \ldots, x_n$ be in $\mathbb{R}^m$ and $c_1, c_2, \ldots, c_n$ in $\mathbb{R}$ satisfying (1.2). Define auxiliary points y_j, $j = 1,2,\ldots,2n$, in the following way:

$$y_j = \begin{cases} x_j, & j = 1,2,\ldots,n, \\ r x_{j-n}, & j = n+1, n+2, \ldots, 2n, \end{cases} \tag{3.24}$$

and corresponding scalers d_j^μ in the form

$$d_j^\mu = \begin{cases} c_j, & \text{if } j = 1,2,\ldots,n, \\ (-1)^\mu c_{j-n}, & \text{if } j = n+1, n+2, \ldots, 2n. \end{cases} \tag{3.25}$$

If $\mathcal{P}$ is homogeneous, we can deduce that

$$\sum_{j=1}^{2n} d_j^\mu p(y_j) = \sum_{j=1}^{n} c_j p(x_j) + (-1)^\mu \sum_{j=1}^{n} c_j p(r x_j) = 0, \quad p \in \mathcal{P}. \tag{3.26}$$

If the kernel $(x,y) \in \mathbb{R}^m \times \mathbb{R}^m \mapsto f(x \cdot y)$ is conditionally positive definite with respect to $\mathcal{P}$, we have that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j f_{r,\mu}(x_i \cdot x_j) = \sum_{i=1}^{2n}\sum_{j=1}^{2n} d_i^{\mu} d_j^{\mu} f(y_i \cdot y_j) \geqslant 0. \tag{3.27}$$

This completes the proof. □

The following result is a consequence of Proposition 3.9.

Proposition 3.10. *Let $\mathcal{P}$ be a homogeneous subspace of Π^m. A kernel $(x, y) \in \mathbb{R}^m \times \mathbb{R}^m \mapsto f(x \cdot y)$ is conditionally positive definite with respect to $\mathcal{P}$ if and only if its odd part $(x, y) \in \mathbb{R}^m \times \mathbb{R}^m \mapsto f(x \cdot y) - f(-x \cdot y)$ and its even part $(x, y) \in \mathbb{R}^m \times \mathbb{R}^m \mapsto f(x \cdot y) + f(-x \cdot y)$ are both conditionally positive definite with respect to $\mathcal{P}$.*

Theorem 3.11. *Let $\mathcal{B}$ be a finite subset of Π^m and $\mathcal{Q} \subset \Pi^{m+l}$ a homogeneous $\mathcal{B}$-dilation. Put $\mathcal{P} = [\mathcal{B}]$ and let β be as before. If f is $\mathbb{C}^{\beta+1}$ and $(x, y) \in \mathbb{R}^{m+l} \times \mathbb{R}^{m+l} \mapsto f(x \cdot y)$ is conditionally positive definite with respect to $\mathcal{Q}$ then f is an everywhere convergent series of the form*

$$f(t) = \sum_{k=0}^{\infty} a_k(f) t^k, \quad a_k(f) \geqslant 0,\ k \geqslant \beta + 1. \tag{3.28}$$

Proof. Under the given conditions, if f is $\mathbb{C}^{\beta+1}$ and $(x, y) \in \mathbb{R}^{m+l} \times \mathbb{R}^{m+l} \mapsto f(x \cdot y)$ is conditionally positive definite with respect to $\mathcal{Q}$, Theorem 3.7 reveals that $f^{(\beta+1)}$ is absolutely monotonic in $[0, \infty)$. In particular,

$$f^{(\beta+1)}(t) = \sum_{k=0}^{\infty} \frac{f^{(\beta+1+k)}(0^+)}{(\beta+1+k)!} t^k, \quad 0 \leqslant t < \infty, \tag{3.29}$$

in which $f^{(\beta+1+k)}(0^+) \geqslant 0$, $k \geqslant 0$. By Proposition 3.10, the odd part of the kernel is conditionally positive definite with respect to $\mathcal{Q}$. Thus, the same trick applied to this kernel leads to

$$\begin{aligned} &f^{(\beta+1)}(t) - (-1)^{\beta+1} f^{(\beta+1)}(-t) \\ &\quad = \sum_{k=0}^{\infty} \frac{f^{(\beta+1+k)}(0^+) - (-1)^{\beta+1+k} f^{(\beta-1+k)}(0^+)}{(\beta+1+k)!} t^k, \quad 0 \leqslant t < \infty. \end{aligned}$$

Combining with (3.29), we obtain

$$f^{(\beta+1)}(-t) = \sum_{k=0}^{\infty} \frac{(-1)^k f^{(\beta+1+k)}(0^+)}{(\beta+1+k)!} t^k, \quad 0 \leqslant t < \infty, \tag{3.30}$$

that is, (3.29) holds for $t \in \mathbb{R}$. Integration leads to the following representation for f:

$$\begin{aligned} f(t) &= a_0(f) + a_1(f)t + \cdots + a_\beta(f)t^\beta + \sum_{k=\beta+1}^{\infty} \frac{f^{(k)}(0^+)}{k!} t^k \\ &= a_0(f) + a_1(f)t + \cdots + a_\beta(f)t^\beta + \sum_{k=\beta+1}^{\infty} \frac{f^{(k)}(0)}{k!} t^k, \quad t \in \mathbb{R}, \end{aligned}$$

in which $f^{(k)}(0) \geqslant 0$, $k = \beta + 1, \beta + 2, \ldots.$ □

Combining Theorems 3.4 and 3.11, we obtain:

Corollary 3.12. *Let $\mathcal{B}$ be a finite subset of Π^m and $\mathcal{Q} \subset \Pi^{m+l}$ a homogeneous $\mathcal{B}$-dilation. Put $\mathcal{P} = [\mathcal{B}]$ and let β be as before. If f is $\mathbb{C}^{\beta+1}$ and $(x, y) \in \mathbb{R}^{m+l} \times \mathbb{R}^{m+l} \mapsto f(x \cdot y)$ is conditionally positive definite with respect to $\mathcal{Q}$ then f is an everywhere convergent series of the form*

$$f(t) = \sum_{k=0}^{\infty} a_k(f) t^k, \quad a_k(f) \geqslant 0,\ k \in \Gamma^m_{[\mathcal{B}]}. \tag{3.31}$$

It would be desirable to have the condition on the coefficients of f in the previous theorem holding for $k \in \Gamma^{m+l}_{\mathcal{Q}}$. Since the inclusion $\Gamma^m_{[\mathcal{B}]} \subset \Gamma^{m+l}_{\mathcal{Q}}$ can be proper, additional hypotheses may be needed in order to change the condition.

Theorem 3.13. *Let f be a $\mathbb{C}^{\beta+1}$ function, $\mathcal{B}$ be a finite subset of Π^m and $\mathcal{Q} \subset \Pi^{m+l}$ a homogeneous and adequate $\mathcal{B}$-dilation. Assume that $\Gamma^{m+l}_{\mathcal{Q}} \cap \{0, 1, \ldots, \beta\} = \emptyset$. Then $(x, y) \in \mathbb{R}^{m+l} \times \mathbb{R}^{m+l} \mapsto f(x \cdot y)$ is conditionally positive definite with respect to $\mathcal{Q}$ if and only if f is an everywhere convergent series of the form*

$$f(t) = \sum_{k=0}^{\infty} a_k(f) t^k, \quad a_k(f) \geqslant 0,\ k \in \Gamma^{m+l}_{\mathcal{Q}}. \tag{3.32}$$

Proof. One implication follows from Corollary 3.12 and our hypothesis on $\Gamma^{m+l}_{\mathcal{Q}}$. As for the other one may proceed as in the proof of Theorem 3.3. $\square$

Example ($\boldsymbol{m + l = 3}$). The space

$$\mathcal{Q} = \big[\big\{1, x, y, z, x^2, y^2, z^2, xz, yz, xy\big\}\big] \tag{3.33}$$

is a homogeneous $\mathcal{B}$-dilation, in which $\mathcal{B} = \{1, x, y, x^2, y^2, xy\}$. Since $\Gamma^3_{\mathcal{Q}} = \{3, 4, \ldots\}$, $\mathcal{Q}$ is adequate and $\Gamma^3_{\mathcal{Q}} \cap \{0, 1, 2\} = \emptyset$. Theorem 3.13 is then applicable in this case.

We like to think that Theorem 3.13 can be refined somehow. There is a chance that the differentiability condition on the function can be weakened. That would require an adaptation of Theorem 1.2 in [6] and perhaps the main result in [2]. We intend to investigate this question in a subsequent work.

4. Strictly conditionally positive definite kernels

In this section, we fix a certain finite-dimensional subspace $\mathcal{P}$ of Π^m and a function f fitting the description produced by Theorem 3.3 and search for necessary and sufficient conditions in order that the dot product kernel on $\mathbb{R}^m$ generated by f is strictly conditionally positive definite with respect to $\mathcal{P}$. The results provide a description of a large class of strictly conditionally positive definite kernels on $\mathbb{R}^m$ with respect to $\mathcal{P}$. It becomes a complete characterization of conditional positive definiteness when the polynomial space has some special features as described in Theorem 3.13.

We begin with a basic necessary condition for strict conditional positive definiteness that does not depend on the representation of f.

Proposition 4.1. *Let A be a subset of $\mathbb{R}^m$ and $\mathcal{P}$ a subspace of Π^m. If the kernel $(x,y) \in A \times A \mapsto f(x \cdot y)$ is conditionally positive definite with respect to $\mathcal{P}$ then $f(y \cdot y) \geqslant 0$, $y \in \{x \in A\colon\ p(x) = 0,\ p \in \mathcal{P}\}$. If it is strictly conditionally positive definite then $f(y \cdot y) > 0$, $y \in \{x \in A\colon\ p(x) = 0,\ p \in \mathcal{P}\}$.*

Proof. Let $y \in \{x \in A\colon\ p(x) = 0,\ p \in \mathcal{P}\}$. Take $n = 1$, $c_1 = 1$ and $x_1 = y$ in the definition of conditional positive definiteness to immediately obtain $f(y \cdot y) \geqslant 0$. The other part is similar. □

Remark. Let $(x,y) \in A \times A \mapsto f(x \cdot y)$ be conditionally positive definite with respect to $\mathcal{P}$. If $f(x \cdot x) > 0$, $x \in A$, then the case $n = 1$ in the definition of strict conditional positive definiteness may be disregarded. Indeed, the condition $cf(x \cdot x)c = 0$ always implies $c = 0$.

Given a polynomial space $\mathcal{P}$, all the results in the rest of the section refer to a fixed function f having an everywhere convergent series of the form

$$f(t) = \sum_{k=0}^{\infty} a_k(f) t^k, \quad a_k(f) \geqslant 0,\ k \in \Gamma_{\mathcal{P}}^m. \tag{4.1}$$

For such a function

$$K_{\mathcal{P}}(f) := \Gamma_{\mathcal{P}}^m \cap \big\{k\colon\ a_k(f) > 0\big\}. \tag{4.2}$$

If $K \subset \mathbb{Z}_+$ we put

$$K_e := K \cap 2\mathbb{Z}_+ \quad \text{and} \quad K_o := K \cap (2\mathbb{Z}_+ + 1). \tag{4.3}$$

Theorem 4.2. *Let A be an infinite and symmetric subset of $\mathbb{R}^m$ and $\mathcal{P}$ a finite-dimensional subspace of Π^m. If f is as in (4.1) and $(x,y) \in A \times A \mapsto f(x \cdot y)$ is strictly conditionally positive definite with respect to $\mathcal{P}$ then both $K_{\mathcal{P}}(f)_e$ and $K_{\mathcal{P}}(f)_o$ are infinite.*

Proof. We show that if $K_{\mathcal{P}}(f)_e$ is finite then $(x,y) \in A \times A \mapsto f(x \cdot y)$ is not strictly conditionally positive definite with respect to $\mathcal{P}$. Define

$$L := \big\{\alpha \in \mathbb{Z}_+^m\colon\ |\alpha| \in K_{\mathcal{P}}(f)_e\big\} \cup \big\{\alpha \in \mathbb{Z}_+^m\colon\ |\alpha| \leqslant \beta\big\} \tag{4.4}$$

and let l denote its cardinality. Take $l + 1$ distinct points $x_1, x_2, \ldots, x_{l+1}$ in $A \setminus \{0\}$ such that no two of them are antipodal. Choose a nonzero solution $(c_1, c_2, \ldots, c_{l+1})$ of the system

$$\sum_{i=1}^{l+1} c_i x_i^\alpha = 0, \quad \alpha \in L. \tag{4.5}$$

It is easily seen that

$$\sum_{i=1}^{l+1} c_i x_i^\alpha + \sum_{i=1}^{l+1} c_i (-x_i)^\alpha = \begin{cases} 2\sum_{i=1}^{l+1} c_i x_i^\alpha, & |\alpha| \in 2\mathbb{Z}, \\ 0, & \text{otherwise}. \end{cases} \tag{4.6}$$

Writing

$$y_i = \begin{cases} x_i, & i = 1, 2, \ldots, l+1, \\ -x_{i-l-1}, & i = l+2, l+3, \ldots, 2l+2, \end{cases} \tag{4.7}$$

and

$$d_i = \begin{cases} c_i, & i = 1, 2, \dots, l+1, \\ c_{i-l-1}, & i = l+2, l+3, \dots, 2l+2, \end{cases} \tag{4.8}$$

we have that

$$\begin{aligned}\sum_{i=1}^{2l+2}\sum_{j=1}^{2l+2} d_i d_j f(y_i \cdot y_j) &= \sum_{k=0}^{\infty} a_k(f) \sum_{i=1}^{2l+2}\sum_{j=1}^{2l+2} d_i d_j \left(\sum_{|\alpha|=k} \frac{k!}{\alpha!} y_i^\alpha y_j^\alpha \right) \\ &= \sum_{k=0}^{\infty} a_k(f) \sum_{|\alpha|=k} \frac{k!}{\alpha!} \left(\sum_{i=1}^{2l+2} d_i y_i^\alpha \right)^2 \\ &= \sum_{k=0}^{\infty} a_k(f) \sum_{|\alpha|=k} \frac{k!}{\alpha!} \left(\sum_{i=1}^{l+1} c_i x_i^\alpha + \sum_{i=1}^{l+1} c_i (-x_i)^\alpha \right)^2.\end{aligned}$$

We show the above sum is zero by looking at every inner sum. Let k be such that $a_k(f) \neq 0$. If $k \in K_{\mathcal{P}}(f)_o$ the corresponding summand above is zero due to (4.6) while if $k \in K_{\mathcal{P}}(f)_e$ it is zero by our choice of the c_j and the x_j and the definition of L. If $k \notin K_{\mathcal{P}}(f)$ only the subcase $k \notin \Gamma_{\mathcal{P}}^m$ deserves attention. Since this implies $k \leqslant \beta$, the summand is zero by (4.5). Thus, the previous sum is indeed zero. Since the y_i are distinct elements of $\mathbb{R}^m$, $\sum_{1=1}^{2l+2} |d_i| > 0$ and $\sum_{i=1}^{2l+2} d_i p(y_i) = 0$, $p \in \mathcal{P}$, we have reached a contradiction to the fact that $(x, y) \in A \times A \mapsto f(x \cdot y)$ is strictly conditionally positive definite with respect to $\mathcal{P}$. If $K_{\mathcal{P}}(f)_o$ has finitely many elements, a minor modification of the above procedure leads to the same conclusion. □

Next, we go the other direction around, first analyzing the case $m = 1$.

Theorem 4.3. *Let f be as in* (4.1)*, A a subset of $\mathbb{R} \setminus \{0\}$ and $\mathcal{P}$ an adequate subspace of Π^1. If both $K_{\mathcal{P}}(f)_e$ and $K_{\mathcal{P}}(f)_o$ are infinite then $(x, y) \in A \times A \mapsto f(x \cdot y)$ is strictly conditionally positive definite with respect to $\mathcal{P}$.*

Proof. Let n be a positive integer, $x_1, x_2, \dots, x_n$ distinct elements of A and $c_1, c_2, \dots, c_n$ real numbers satisfying (1.2). We show that, under the given hypotheses, the equality

$$Q := \sum_{i,j=1}^{n} c_i c_j f(x_i x_j) = 0 \tag{4.9}$$

implies $c_i = 0$, $i = 1, 2, \dots, n$. As explained before, we can write

$$Q = \sum_{k \in K_{\mathcal{P}}(f)} a_k(f) \left(\sum_{i=1}^{n} c_i x_i^k \right)^2. \tag{4.10}$$

The condition $Q = 0$ implies that

$$\sum_{i=1}^{n} c_i x_i^k = 0, \quad k \in K_{\mathcal{P}}(f). \tag{4.11}$$

Since the case $n = 1$ is trivial, we assume $n > 1$. Let x_1 have maximum modulus among the x_i. Defining $y_i := x_i / x_1$, $i = 1, 2, \dots, n$, Eq. (4.11) reduces to

$$\sum_{|y_i|=1} c_i y_i^k + \sum_{|y_i|<1} c_i y_i^k = 0, \quad k \in K_{\mathcal{P}}(f). \tag{4.12}$$

If $n = 2$ and $y_2 = -1$, our assumptions on $K_{\mathcal{P}}(f)$ imply that $c_1 - c_2 = c_1 + c_2 = 0$, that is, $c_1 = c_2 = 0$. Otherwise, given $\epsilon > 0$ there is an index $N = N(\epsilon) > 0$ such that

$$\left| \sum_{|y_i|<1} c_i y_i^k \right| < \epsilon, \quad k \in K_{\mathcal{P}}(f),\ k \geqslant N, \tag{4.13}$$

whence

$$\left| \sum_{|y_i|=1} c_i y_i^k \right| < \epsilon, \quad k \in K_{\mathcal{P}}(f),\ k \geqslant N. \tag{4.14}$$

It follows that either

$$|c_1| < \epsilon, \quad k \in K_{\mathcal{P}}(f),\ k \geqslant N, \tag{4.15}$$

or

$$\left| c_1 + c_j(-1)^k \right| < \epsilon, \quad k \in K_{\mathcal{P}}(f),\ k \geqslant N, \tag{4.16}$$

for some j. Since ϵ is arbitrary, we conclude that $c_1 = 0$ in the first case and $c_1 = c_j = 0$ in the second one. After eliminating these two c_i in (4.11) we can re-apply the process. After finitely many steps, we conclude that all the c_i are zero. □

Theorem 4.4. *Let f be as in* (4.1)*, A a subset of $\mathbb{R}$ and $\mathcal{P}$ an adequate subspace of Π^1. If $f(0) > 0$ and both sets $K_{\mathcal{P}}(f)_e$ and $K_{\mathcal{P}}(f)_o$ are infinite then $(x, y) \in A \times A \mapsto f(x \cdot y)$ is strictly conditionally positive definite with respect to $\mathcal{P}$.*

Proof. Let $x_1, x_2, \ldots, x_n$ be distinct elements of A and $c_1, c_2, \ldots, c_n$ real numbers satisfying (1.2). We will show that $Q > 0$ when at least one c_i is nonzero. Only the case in which 0 is among the x_i deserves attention. Without loss of generality, we can assume $x_1 = 0$. If $n = 1$, $Q = c_1^2 f(0) > 0$. So, we assume $n > 1$ and divide the analysis in two cases. If $c_1 \neq 0$, we write

$$Q = a_0(f)\left(\sum_{i=1}^{n} c_i\right)^2 + \sum_{\substack{k \in K_{\mathcal{P}}(f) \\ k \neq 0}} a_k(f)\left(\sum_{i=1}^{n} c_i x_i^k\right)^2. \tag{4.17}$$

Then, if $c_i = 0$, $i \neq 1$, $Q = f(0)c_1^2 > 0$. Otherwise, the second sum above is positive by the description of strict conditional positive definiteness with respect to $\{0\}$ given in [13]. If $c_1 = 0$ then

$$\sum_{i=2}^{n} c_i p(x_i) = \sum_{i=1}^{n} c_i p(x_i) = 0, \quad p \in \mathcal{P}, \tag{4.18}$$

and

$$Q = a_0(f)\left(\sum_{i=2}^{n} c_i\right)^2 + \sum_{\substack{k \in K_{\mathcal{P}}(f) \\ k \neq 0}} a_k(f)\left(\sum_{i=2}^{n} c_i x_i^k\right)^2 > 0, \tag{4.19}$$

because the second sum is positive by the previous theorem. □

Now, we move to higher dimensions. The following elementary lemma, essentially proved in [14, p. 287] and quoted in a simpler form in [13], is needed.

Lemma 4.5. *Let B be a real $n \times n$ nonnegative definite matrix with no identical rows. Then there are distinct real numbers $b_1, b_2, \ldots, b_n$ and a nonnegative definite matrix P such that $B_{\mu\nu} = b_\mu b_\nu + P_{\mu\nu}$, $\mu, \nu = 1, 2, \ldots, n$. If no column of B is zero then the b_i can be chosen nonzero.*

Theorem 4.6. *Let f be as in* (4.1)*, A a subset of $\mathbb{R}^m \setminus \{0\}$ and $\mathcal{P}$ an adequate subspace of Π^m. If both sets $K_{\mathcal{P}}(f)_e$ and $K_{\mathcal{P}}(f)_o$ are infinite then $(x, y) \in A \times A \mapsto f(x \cdot y)$ is strictly conditionally positive definite with respect to $\mathcal{P}$.*

Proof. Due to Theorem 4.3, we can assume that $m \geqslant 2$. Let n be a positive integer, $x_1, x_2, \ldots, x_n$ distinct elements of A and $c_1, c_2, \ldots, c_n$ real numbers satisfying $\sum_{i=1}^n |c_i| > 0$ and (1.2). We will show that the corresponding quadratic form Q is positive. We can assume $n > 1$. We use the previous lemma to write

$$x_i \cdot x_j = b_i b_j + P_{ij}, \quad i, j = 1, 2, \ldots, n, \tag{4.20}$$

in which $b_i \in \mathbb{R} \setminus \{0\}$, $i = 1, 2, \ldots, n$, $b_i \neq b_j$, $i \neq j$, and P is an $n \times n$ nonnegative definite matrix. Since

$$\begin{aligned} Q &= \sum_{k \in K_{\mathcal{P}}(f)} a_k(f) \sum_{i=1}^n \sum_{j=1}^n c_i c_j (x_i \cdot x_j)^k \\ &= \sum_{k \in K_{\mathcal{P}}(f)} a_k(f) \sum_{\mu+\nu=k} \frac{k!}{\mu!\nu!} \sum_{i=1}^n \sum_{j=1}^n c_i c_j b_i^\mu b_j^\mu P_{ij}^\nu, \end{aligned}$$

it follows that

$$Q \geqslant \sum_{k \in K_{\mathcal{P}}(f)} a_k(f) \sum_{i=1}^n \sum_{j=1}^n c_i c_j b_i^k b_j^k = \sum_{i=1}^n \sum_{j=1}^n c_i c_j g(b_i b_j), \tag{4.21}$$

in which

$$g(t) = \sum_{k \in K_{\mathcal{P}}(f)} a_k(f) t^k. \tag{4.22}$$

The inequality above follows from the fact that each inner quadratic form in the very last sum preceding the inequality is nonnegative. The nonnegativity of the forms is a consequence of the Schür product theorem [7, p. 455]. Due to our assumptions, the kernel $(x, y) \in (\mathbb{R} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\}) \mapsto g(x \cdot y)$ is strictly conditionally positive definite with respect to $\{0\}$. Thus $Q > 0$. $\square$

Corollary 4.7. *Let f be as in* (4.1)*, A a subset of $\mathbb{R}^m$ and $\mathcal{P}$ an adequate subspace of Π^m. If $f(0) > 0$ and both sets $K_{\mathcal{P}}(f)_e$ and $K_{\mathcal{P}}(f)_o$ are infinite then $(x, y) \in A \times A \mapsto f(x \cdot y)$ is strictly conditionally positive definite with respect to $\mathcal{P}$.*

The following characterizations are now evident.

Theorem 4.8. *Let f be as in* (4.1)*, A an infinite and symmetric subset of $\mathbb{R}^m \setminus \{0\}$ and $\mathcal{P}$ a finite-dimensional adequate subspace of Π^m. Then $(x, y) \in A \times A \mapsto f(x \cdot y)$ is strictly conditionally positive definite with respect to $\mathcal{P}$ if and only if $K_{\mathcal{P}}(f)_e$ and $K_{\mathcal{P}}(f)_o$ are both infinite.*

Theorem 4.9. *Let f be as in* (4.1)*, A an infinite and symmetric subset of $\mathbb{R}^m$ and $\mathcal{P}$ a finite-dimensional adequate subspace of Π^m. Then $(x, y) \in A \times A \mapsto f(x \cdot y)$ is strictly conditionally positive definite with respect to $\mathcal{P}$ if and only if $f(0) > 0$ and $K_{\mathcal{P}}(f)_e$ and $K_{\mathcal{P}}(f)_o$ are both infinite.*

We end the section by observing that the last theorem in Section 3 and the results in this section do not produce a characterization for strict conditional positive definiteness yet, even if the polynomial space $\mathcal{Q}$ has the conditions stated in Theorem 3.13. Indeed, we do not know whether a conditionally positive definite with respect to $\mathcal{Q}$ is differentiable up to order $\beta + 1$.

5. Additional results

If we consider a more restrictive class of functions, then it is reasonable to expect the results in Section 4 to hold for more general polynomial spaces. In this section we indicate a way of doing this.

Throughout this section, if $\mathcal{P}$ is a polynomial space, g will be a fixed function having an everywhere convergent series representation in the form

$$g(t) = \sum_{k=0}^{\infty} a_k(g) t^k, \quad a_k(g) \geqslant 0,\ k \in \Upsilon_{\mathcal{P}}^m, \tag{5.1}$$

in which

$$\Upsilon_{\mathcal{P}}^m := \mathbb{Z}_+ \setminus \big\{k\colon x^\alpha \in \mathcal{P},\ \alpha \in \mathbb{Z}_+^m,\ |\alpha| = k\big\}. \tag{5.2}$$

It is very easy to see that $\Gamma_{\mathcal{P}}^m \subset \Upsilon_{\mathcal{P}}^m$. Thus, the condition on the coefficients of g is more restrictive than the condition imposed on the function f in Section 4. If we define

$$K'_{\mathcal{P}}(g) := \Upsilon_{\mathcal{P}}^m \cap \big\{k\colon a_k(g) > 0\big\} \tag{5.3}$$

and proceed as in the proof of Theorem 4.2, we obtain the following result.

Theorem 5.1. *Let g be as in* (5.1)*, $\mathcal{P}$ a finite-dimensional subspace of Π^m and A an infinite and symmetric subset of $\mathbb{R}^m$. If $(x, y) \in A \times A \mapsto g(x \cdot y)$ is strictly conditionally positive definite with respect to $\mathcal{P}$ then both $K'_{\mathcal{P}}(g)_e$ and $K'_{\mathcal{P}}(g)_o$ are infinite.*

All remaining theorems in Section 4 have their counterpart here, without the adequate assumption on $\mathcal{P}$. Following the proofs of those theorems, the key step in the process of adapting them, is to make sure that the sums associated to the quadratic forms can be indexed by the condition $k \in K'_{\mathcal{P}}(g)$. To see that we can do that, let $k \notin K'_{\mathcal{P}}(g)$. If $k \in \Upsilon_{\mathcal{P}}^m$ then we know that $a_k(g) \geqslant 0$ by the way we set g and we know that $a_k(g) \leqslant 0$ by the definition of $\Upsilon_{\mathcal{P}}^m$. Thus, this case may be disregarded. If $k \notin \Upsilon_{\mathcal{P}}^m$, then the corresponding summand is zero due to the condition the points x_i and the scalars c_i must satisfy, according to the definition of conditionally positive definiteness. This point being resolved, the following theorems follow from those in Section 4.

Theorem 5.2. *Let g be as in* (5.1)*, $\mathcal{P}$ a subspace of Π^m and A a subset of $\mathbb{R}^m \setminus \{0\}$. If both $K'_{\mathcal{P}}(g)_e$ and $K'_{\mathcal{P}}(g)_o$ are infinite then $(x, y) \in A \times A \mapsto g(x \cdot y)$ is strictly conditionally positive definite with respect to $\mathcal{P}$.*

Theorem 5.3. *Let g be as in* (5.1)*, $\mathcal{P}$ a subspace of Π^m and A a subset of $\mathbb{R}^m$. If $g(0) > 0$ and both $K'_{\mathcal{P}}(g)_e$ and $K'_{\mathcal{P}}(g)_o$ are infinite then $(x, y) \in A \times A \mapsto g(x \cdot y)$ is strictly conditionally positive definite with respect to $\mathcal{P}$.*

Theorem 5.4. *Let g be as in* (5.1)*, $\mathcal{P}$ a finite-dimensional subspace of Π^m and A an infinite and symmetric subset of $\mathbb{R}^m \setminus \{0\}$. Then $(x, y) \in A \times A \mapsto g(x \cdot y)$ is strictly conditionally positive definite with respect to $\mathcal{P}$ if and only if $K'_{\mathcal{P}}(g)_e$ and $K'_{\mathcal{P}}(g)_o$ are infinite.*

Theorem 5.5. *Let g be as in* (5.1)*, $\mathcal{P}$ a finite-dimensional subspace of Π^m and A an infinite and symmetric subset of $\mathbb{R}^m$. Then $(x, y) \in A \times A \mapsto g(x \cdot y)$ is strictly conditionally positive definite with respect to $\mathcal{P}$ if and only if $g(0) > 0$ and $K'_{\mathcal{P}}(g)_e$ and $K'_{\mathcal{P}}(g)_o$ are infinite.*

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